

TechNotes Special Edition

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NOTES ON NUMERICAL APPROXIMATIONS FOR PRESSURE TRANSIENT ANALYSIS

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This note presents a collection of notes concerning numerical techniques needed to practically evaluate well tests.

The mathematics of pressure transient analysis requires a multitude of mathematical functions to describe the pressure behavior of the reservoir and the wellbore. In most cases, dimensionless pressure solutions are derived analytically or semi-analytically and involve modified Bessel functions, logarithms, error functions and exponential integrals. In addition, many of the solutions are given in terms of Laplace transforms, which must then be inverted numerically to obtain the true solutions. The accuracy of the analytical solutions is usually fairly easy to evaluate in terms of the assumptions made in the derivation of the equations and the solutions. For practical use, however, it is necessary to evaluate numerically the equations obtained and for modern well test evaluation techniques involving nonlinear regression and graphical user interfaces, it is important that the equations are evaluated both quickly and accurately.

Modified Bessel Functions

The Laplace transform of the basic radial flow equation used in well testing contains modified Bessel functions as shown in Equation 1¹. The line source approximation can be obtained by noting that $xK_1(x)$ approaches 1 as x approaches 0, which corresponds to time approaching infinity. Further, the normal long time approximation can be derived by noting that $K_0(x)$ approaches $-(\ln(x) + \gamma)$ and the inverse Laplace transform yields the normal semilog straight line relationship used in classical analysis methods.

$$L\{p_D\} = \frac{K_0(r_D \sqrt{s})}{s \sqrt{s} K_1(\sqrt{s})} \dots\dots\dots (1)$$

Unfortunately, if we need to represent the dimensionless pressure at early times due to low permeability, high viscosity, a large wellbore or other reasons, it soon becomes apparent that neither the line source approximation nor the long time approximation are valid and a more accurate representation of the modified Bessel functions are needed.

The first approach might be to evaluate the terms in the series expansion for the Bessel functions as given in Abramowitz and Stegun². That turns out to be tedious, slow and prone to numerical roundoff problems. A better approach is to use the numerical approximations presented in Reference 2 or similar approximations

available from other sources. Even so, there comes a point when the approximations are not valid and produce large errors at small values of time, so a better approach should be found.

It is interesting to note that the major problems occur at small values of time, but as a practical matter, the short time response of a well is rarely observed, since it is masked by wellbore storage. We therefore have a quandary in that the Bessel function approximations fall apart and disrupt the entire solution in the part of the solution where they are not physically important. The problem is a result of the fact that both the K0 and K1 modified Bessel functions approach 0 at large values of s and the ratio of the two approximations is almost indeterminate. For that reason, even small errors become important and the precision of the formula degrades rapidly.

However, it is interesting to note that in most cases the individual Bessel functions are not used alone. Instead for an infinite reservoir, they always appear in a ratio as shown in Equation 1. It turns out that it is much easier to approximate the ratio than the individual functions and that the ratio degrades much more gracefully. A formula of the form of Equation 2, where A and n are constants, has been proposed, but it is possible to improve the accuracy by additional terms and various other functional forms.

$$\frac{K_0(x)}{x K_1(x)} \approx \ln\left(1 + \frac{1}{x}\right) + \frac{A}{(1+x)^n} \dots\dots\dots (2)$$

Error Function in Phase Redistribution

Hegeman, et. al.³ defined a phase redistribution function as an alternative to the exponential form originally presented⁴. In general, the phase redistribution function is used in the wellbore equation to account for storage effects and is included in the solution by the use of a convolution integral. Since convolution is a simple product of functions in terms of Laplace transforms¹, the error function itself is not normally required, but its Laplace transform is needed in evaluating the Laplace transform of the wellbore equation. The Hegeman, et. al. phase redistribution function and its Laplace transform are shown in Equation 3.

$$p_{\phi D} = C_{\phi D} \operatorname{erf}\left(\frac{t_D}{\alpha_D}\right)$$

$$L\{p_{\phi D}\} = \frac{C_{\phi D}}{s} \left\{ 1 - e^{\left(\frac{\alpha_D s}{2}\right)^2} \left[1 - \operatorname{erf}\left(\frac{\alpha_D s}{2}\right) \right] \right\} \dots\dots\dots (3)$$

However, Abramowitz and Stegun¹ provide an accurate approximation for the error function in the form of Equation 4, where f(x) is a polynomial. The accuracy of the approximation is stated to be less than 1.5 x 10⁻⁷ over the entire range of 0 to infinity.

$$\operatorname{erf}(x) \approx 1 - f(x) e^{-x^2} \dots\dots\dots (4)$$

Substituting Equation 4 into Equation 3 yields a simplified form for the phase redistribution function Laplace transform as shown in Equation 5. Note that there is no need to evaluate exponentials and the polynomial from the Abramowitz and Stegun approximation can be used directly in the Laplace space wellbore equation, thus saving both computing effort and maintaining precision.

$$L\{p_{\phi D}\} \approx \frac{C_{\phi D}}{s} \left[1 + f\left(\frac{\alpha_D s}{2}\right) \right] \dots\dots\dots (5)$$

Infinite Conductivity and Uniform Flux Vertical Fractures

The pressure solution for these functions are related and given by the general form shown in Equation 6⁵. Various means for representing these and their Laplace transforms were reviewed by Blasingame, et. al. The solution for the well pressure for the uniform flux case is calculated by using x_D = 0 while the infinite conductivity case uses x_D = 0.732. (In the equation, we show the exponential integral as E₁ consistent with Abramowitz and Stegun¹, although reference 5 shows it as E_i.)

$$p_D(x_D, t_D) = \frac{1}{2} \sqrt{\pi t_D} \left[\operatorname{erf} \left(\frac{1-x_D}{2\sqrt{t_D}} \right) + \operatorname{erf} \left(\frac{1+x_D}{2\sqrt{t_D}} \right) \right] + \frac{1-x_D}{4} E_1 \left[\frac{(1-x_D)^2}{4t_D} \right] + \frac{1+x_D}{4} E_1 \left[\frac{(1+x_D)^2}{4t_D} \right] \quad \dots\dots\dots(6)$$

This function can be rearranged to combine terms involving $1 - x_D$ and $1 + x_D$ as shown in Equation 7. In this form it is apparent that we can parameterize both terms by using $\tau_1 = \frac{t_D}{(1-x_D)^2}$, $\tau_2 = \frac{t_D}{(1+x_D)^2}$.

Substituting this yields Equation 8. Of course, when $x_D = 0$, $\tau = t_D$.

$$p_D(x_D, t_D) = \frac{1}{2} \left[\sqrt{\pi t_D} \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{(1-x_D)^2}{t_D}} \right) + \frac{1-x_D}{2} E_1 \left(\frac{1}{4} \frac{(1-x_D)^2}{t_D} \right) \right] + \frac{1}{2} \left[\sqrt{\pi t_D} \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{(1+x_D)^2}{t_D}} \right) + \frac{1+x_D}{2} E_1 \left(\frac{1}{4} \frac{(1+x_D)^2}{t_D} \right) \right] \quad \dots\dots\dots(7)$$

$$p_D(x_D) = \frac{(1-x_D)}{2} \left[\sqrt{\pi \tau_1} \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{\tau_1}} \right) + \frac{1}{2} E_1 \left(\frac{1}{4\tau_1} \right) \right] + \frac{(1+x_D)}{2} \left[\sqrt{\pi \tau_2} \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{\tau_2}} \right) + \frac{1}{2} E_1 \left(\frac{1}{4\tau_2} \right) \right] \quad \dots\dots\dots(8)$$

Consider first the uniform flux case where $x_D = 0$, $\tau = t_D$ and the equation simplifies to Equation 9. Note that there are two terms, an error function and an exponential integral. At early times, the exponential integral is small and the error function is close to 1, so the pressure changes proportional to the square root of time. At long times the error function is small and the exponential integral approaches a logarithm function of time yielding a semilog straight line.

$$p_D(\tau) = \sqrt{\pi \tau} \operatorname{erf} \left(\frac{1}{2\sqrt{\tau}} \right) + \frac{1}{2} E_1 \left(\frac{1}{4\tau} \right) \quad \dots\dots\dots(9)$$

This equation is easily evaluated using the approximations given in Abramowitz and Stegun¹, but that doesn't help a lot because in order to include wellbore effects we need the Laplace transform so we can easily compute the required convolution integral. Unfortunately neither the error function nor the exponential integral have tabulated Laplace transforms. We note, however, that the exponential integral term comes from the line source approximation, which is the same for a radial wellbore, so we can substitute our previous function in terms of modified Bessel functions to compute the Laplace transform of that term.

At long times $1/(2\sqrt{\tau})$ is small and the error function is approximately $1/\sqrt{\pi \tau}$ so the first term approaches 1 and its Laplace transform is $1/s$ as $s \rightarrow 0$. At short times $1/(2\sqrt{\tau})$ becomes large and the error function approaches 1 and the term approaches $\sqrt{\pi \tau}$ with Laplace transform approaching $\pi/(2s\sqrt{s})$ as $s \rightarrow \infty$. It is also interesting to note that both of these terms satisfy exactly an approximation similar to that of Schapery⁷ as reported by Blasingame, et. al.⁶ which says that a function is approximately equal to the Laplace transform of its derivative evaluated at C/t , where C is a constant. For this case, evaluating the function $\pi/(2\sqrt{s})$ at $s = \pi/(4t)$ yields the exact short time approximation. Since we know that the approximation is exact for the long and short time approximations, it is reasonable to hope that it might also hold for the composite function representing the error function term.

If the approximation is close enough, then we can propose that the complete transform of Equation 7 is as shown in Equation 10.

$$L\{p_D(\tau)\} \approx \frac{\pi}{2s\sqrt{s}} \operatorname{erf}\left(\sqrt{\frac{s}{\pi}}\right) + \frac{K_0(\sqrt{s})}{s\sqrt{s}K_1(\sqrt{s})} \dots\dots\dots (10)$$

Once we know how to obtain the Laplace transform for the uniform flux case, the infinite conductivity case is relatively easy. Using the identity $L\left\{\frac{1}{c}F\left(\frac{t}{c}\right)\right\} = f(cs)$ we can evaluate the function at times indicated by the expressions for τ_1 and τ_2 , multiply by the respective coefficients and sum the results.

Conclusions

In conclusion, even though the equations for well testing are fairly well known and understood, there are still challenges in evaluating the various functions needed in the numerical calculations. This note has presented some considerations, however, it is always useful to review all calculations for numerical precision and computing speed.

Nomenclature

Defined in text

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