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A NOTE ON THE NUMERICAL CALCULATION OF INFINITE INTEGRALS

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In numerical computations, it is sometimes necessary to calculate the value of an infinite integral, i.e. an integral with a limit of infinity. The calculation of integrals is fairly straightforward, using the trapezoidal formula or Simpson's rule, but when one or both limits are infinity, it presents a challenge for computer implementation. One example of an infinite integral is the numerical computation of Laplace transforms, sometimes needed in pressure transient analysis.

To compute the value of an integral with a limit of infinity, a variable transformation can be used to convert the infinite limit to a finite limit of zero. Consider the integral

$$y=\int_a^{\infty} F(x)\,dx$$

This integral can be rewritten using a change of variables as follows:

$$Let\ x=1/u\,,\ dx=-\frac{1}{u^2}du$$

$$y=\int_{1/a}^{0} -\frac{1}{u^2}F\left(\frac{1}{u}\right)du$$

$$y=\int_{0}^{1/a} \frac{1}{u^2}F\left(\frac{1}{u}\right)du$$

The new integral has no infinite limits and can be evaluated using standard numerical methods, i.e. the trapezoidal rule or Simpson's rule.

For the specific case of the Laplace transform integral, the integral can be split into two integrals, the variable transformed, and the resulting integrals can be recombined as follows:

$$f(s)=\int_0^{\infty} e^{-st}F(t)\,dt=\int_0^{1} e^{-st}F(t)\,dt+\int_1^{\infty} e^{-st}F(t)\,dt$$

$$f(s)=\int_0^{1} e^{-st}F(t)\,dt+\int_0^{1} \frac{e^{-s/t}}{t^2}F\left(\frac{1}{t}\right)dt$$

and therefore the Laplace transform integral can be written as a simple definite integral as follows:

$$f(s)=\int_0^1 \left[e^{-st} F(t) + \frac{e^{-s/t}}{t^2} F\left(\frac{1}{t}\right) \right] dt$$

It should be noted that as $t \to 0,$ the second term in the integral approaches zero if the Laplace transform exists, thereby simplifying the calculation. Depending on the nature of the function to be integrated, it may be possible to simplify the relationship even more.

As a side note, experience indicates that greater than 6 digit accuracy can normally be achieved using Simpson's rule with about 10,000 terms and extended precision arithmetic for most functions.